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M-theory compactified on Calabi-Yau fourfolds with background flux[1]

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ABSTRACT

We perform the Kaluza-Klein reduction of M-theory on warped Calabi-Yau fourfolds with non-trivial four-form flux turned on. The resulting scalar- and superpotential is computed and compared with the superpotential obtained by Gukov, Vafa and Witten using different methods.

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String vacua with 4 unbroken supercharges are of particular interest due to their possible phenomenological properties. In four space-time dimensions such vacua have traditionally been constructed as compactifications of the heterotic string on Calabi-Yau threefolds Y_3 or conformal field theoretic generalizations thereof. More recently other classes of vacua with four supercharges have been considered such as F-theory compactified on elliptically fibered Calabi-Yau fourfolds Y_4 [1].

In this paper we study M-theory compactified on Y_4 whose low energy effective theory has three space-time dimensions and also four unbroken supercharges. Aspects of such compactifications have been considered in refs. [2–9] and in an appropriate limit they are related to F-theory on Y_4 [1]. In a previous paper [7] we derived the low energy effective Lagrangian in the large volume limit for the class of fourfolds with vanishing Euler number $\chi = 0$. In particular this constraint implied that the metric is a direct product $M_3 \times Y_4$ and no background flux had to be turned on. In this paper we generalize the previous analysis to the case $\chi \neq 0$ which requires either space-time filling membranes or a non-trivial four-form flux F_4 on Y_4 and a warped space-time metric [2]. We show that a non-trivial F_4 introduces Chern-Simons terms and a potential which in turn can be derived from two superpotentials. The presence of the Chern-Simons terms has first been noticed in ref. [4] while the superpotentials have been proposed in [9]. We find basic agreement with ref. [9] except that one of our superpotentials is the real version of the corresponding superpotential given in [9].[4] Similar superpotentials arise in the compactification of type IIA on Y_4 [11] and their properties have been studied in refs. [9, 11–13].[5]

Let us first briefly recall the compactification of M-theory on a fourfold Y_4 with $\chi(Y_4) = 0$. For this situation the low energy effective Lagrangian in the large volume limit has been derived in ref. [7]. The starting point is the eleven-dimensional supergravity which only contains a metric and a three-form A_3 (with field strength $F_4 = dA_3$) as massless bosonic components. The 11-dimensional supergravity action is given by [15]

$$\mathcal{S} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-\hat{g}} \left[\hat{R} - \frac{1}{2}|F_4|^2 \right] - \frac{1}{12\kappa_{11}^2} \int A_3 \wedge F_4 \wedge F_4 \ . \tag{1}$$

When compactified on a Calabi-Yau fourfold additional scalar and vector fields appear in the resulting three-dimensional effective action. From the metric $h^{1,1}$ real scalar fields $M^A, A = 1, \dots, h^{1,1}$, which parameterize the deformation of the Kähler form and $h^{3,1}$ complex scalar fields $Z^\alpha, \alpha = 1, \dots, h^{3,1}$, which parameterize the deformation of the complex structure arise. In addition the three-form A_3 leads to $h^{1,1}$ vector fields A^A_μ and $h^{2,1}$ complex scalar fields $N^I, I = 1, \dots, h^{2,1}$. Together these fields form $h^{2,1} + h^{3,1}$ chiral multiplets and $h^{1,1}$ vector multiplets. For simplicity we consider only the case where the $(2,1)$-scalars N^I are frozen and perform a Kaluza-Klein reduction keeping only the massless $(1,1)$ and $(3,1)$ modes. In the large volume limit this results in [7]

$$\begin{aligned} \mathcal{L}_0^{(3)} &= \frac{1}{2} R^{(3)} - G_{\alpha\bar{\beta}} \partial_\mu Z^\alpha \partial^\mu \bar{Z}^{\bar{\beta}} - \frac{1}{4} \mathcal{V}^2 G_{AB} F^A_{\mu\nu} F^{B\mu\nu} \\ &\quad - \frac{1}{2} G_{AB} \partial_\mu M^A \partial^\mu M^B - \frac{1}{2} \partial_\mu \ln \mathcal{V} \, \partial^\mu \ln \mathcal{V} \ . \end{aligned} \tag{2}$$

[4]There is another way to generate a superpotential in three dimensions by wrapping 5-branes over certain 6-cycles of Y_4 [3]. This can however not occur if there is a non-vanishing four-form flux localized on a four-dimensional submanifold of the 6-cycle [10]. Such contributions are not consider here.

[5]An expanded version of our paper can be found in [14].

$G_{\alpha\bar\beta}$ is the Kähler metric on the space of (3,1)-forms given by [16]

$$G_{\alpha\bar\beta} \;=\; -e^{K_{3,1}}\int_{Y_4}\Phi_\alpha\wedge\bar\Phi_{\bar\beta} \;=\; \partial_\alpha\bar\partial_{\bar\beta}K_{3,1}\ , \qquad K_{3,1} = -\ln\Big[\int_{Y_4}\Omega\wedge\bar\Omega\Big]\ , \tag{3}$$

where Ω is the unique holomorphic (4,0)-form on Y_4, $\bar\Omega$ its complex conjugate and Φ_α a basis of $H^{3,1}(Y_4)$. G_{AB} is the metric on the space of (1,1)-forms defined as [17]

$$G_{AB} = \frac{1}{2\mathcal{V}}\int_{Y_4} e_A\wedge\star e_B = -\frac{1}{2}\partial_A\partial_B\ln\mathcal{V}\ , \tag{4}$$

where $\mathcal{V}$ is the volume of Y_4

$$\mathcal{V} = \int_{Y_4} d^8\xi\sqrt{g} = \frac{1}{4!}\int_{Y_4} J\wedge J\wedge J\wedge J = \frac{1}{4!}d_{ABCD}M^AM^BM^CM^D\ . \tag{5}$$

$J = M^Ae_A$ is the Kähler form and $d_{ABCD} = \int_{Y_4} e_A\wedge e_B\wedge e_C\wedge e_D$ are the classical intersection numbers of Y_4.

The Lagrangian (2) can also be displayed in dual variables where one dualizes the vectors A^A_μ to scalars P^A and defines the complex Kähler coordinates

$$T^A = \frac{1}{\sqrt{8}}(iP^A + \mathcal{V}G_{AB}M^B)\ . \tag{6}$$

In this basis the Lagrangian (2) reads [7]

$$\mathcal{L}^{(3)}_0 = \frac{1}{2}R^{(3)} - G_{\alpha\bar\beta}\,\partial_\mu Z^\alpha\partial^\mu\bar Z^{\bar\beta} - G^K_{A\bar B}\,\partial_\mu T^A\partial^\mu\bar T^{\bar B}\ , \tag{7}$$

where the metric $G^K_{A\bar B}$ is Kähler and given by

$$G^K_{A\bar B} = \partial_A\bar\partial_{\bar B}K_{1,1}\ , \qquad K_{1,1} = -3\ln\mathcal{V} = -\ln\Big[(T^A+\bar T^A)\mathcal{V}G^{-1}_{AB}(T^B+\bar T^B)\Big]\ . \tag{8}$$

Now we come to the main objective of the paper, that is generalize the previous analysis to the case where $\chi\neq 0$. In the absence of space-time filling membranes an A_3-tadpole can only be avoided in the $D=3$ theory if a non-trivial four-form flux F_4 along the internal Calabi-Yau fourfold is turned on [2, 18, 19]. This arises due to the presence of the higher order term (in κ_{11}) [20]

$$\delta\mathcal{S}_1 = -T_2\int A_3\wedge X_8\ , \tag{9}$$

where $T_2 \equiv (2\pi)^{2/3}(2\kappa_{11}^2)^{-1/3}$ and

$$X_8 = \frac{1}{192\,(2\pi)^4}\left[\mathrm{tr}\hat R^4 - \frac{1}{4}(\mathrm{tr}\hat R^2)^2\right]\ , \qquad \int_{Y_4} X_8 \;=\; -\frac{\chi}{24}\ . \tag{10}$$

Combining eqs. (1), (9), (10) results in absence of space-time filling membranes in the consistency condition [2, 18, 19]

$$\frac{1}{4\kappa_{11}^2}\int_{Y_4} F_4\wedge F_4 \;=\; \frac{T_2}{24}\,\chi\ , \tag{11}$$

which requires a non-vanishing (internal) F_4 if $\chi \neq 0$. Furthermore, supersymmetry can only be maintained if the metric is not a direct product $M_3 \times Y_4$ but instead includes a warp factor Δ [2, 21, 22]

$$\hat{g}_{MN} = \begin{pmatrix} e^{-\Delta} g_{\mu\nu} & 0 \\ 0 & e^{\frac{1}{2}\Delta} g_{mn} \end{pmatrix} , \tag{12}$$

where to leading order in κ_{11} g_{mn} is a Ricci-flat Calabi-Yau metric. The warp factor Δ obeys [2]

$$\nabla_m \partial^m \Delta = -\frac{1}{3} \star (F_4 \wedge F_4) - \frac{4}{3} T_2 \kappa_{11}^2 \star X_8 , \tag{13}$$

where the Laplace operator and the Hodge $\star$-operator are defined with respect to the metric g_{mn}.[6] Thus for compactifications with $\chi \neq 0$ higher order terms have to be taken into account and they in turn warp the three-dimensional space-time metric.

The Kaluza-Klein reduction is a good approximation if the size of the internal Y_4 manifold is large compared to the 11-dimensional Planck length $l_{11}^9 = \kappa_{11}^2$ or in other words for $l_Y \gg l_{11}$ where l_Y^8 is the 'average' size of Y_4. From eq. (11) we infer that $F_4 \sim \mathcal{O}(l_{11}^3/l_Y^4)$ while eq. (13) implies $\Delta \sim \mathcal{O}(l_{11}^6/l_Y^6)$ so that in the limit $l_{11}/l_Y \to 0$ the metric (12) becomes the unwarped product metric and F_4 vanishes [8].[7]

In this paper we focus on the three-dimensional effective theory with at most two derivatives and compute some of the corrections to the Lagrangian (2) which result from higher order terms. Specifically we compute the Chern-Simons term and the potential to order $\mathcal{O}(\kappa_{11}^{-2/3})$ while the corrections to the kinetic terms of (2) are not calculated. With this restriction the only other 11-dimensional term we need to consider is[8]

$$\delta \mathcal{S}_2 = b_1 T_2 \int d^{11}x \sqrt{-g} (J_0 - \frac{1}{2} E_8) , \tag{14}$$

where $b_1^{-1} \equiv (2\pi)^4 3^2 2^{13}$ and

$$\begin{aligned} E_8 &= \frac{1}{3!} \epsilon^{ABCM_1N_1 \ldots M_4N_4} \epsilon_{ABCM'_1N'_1 \ldots M'_4N'_4} \hat{R}^{M'_1N'_1}{}_{M_1N_1} \ldots \hat{R}^{M'_4N'_4}{}_{M_4N_4} , \\ J_0 &= t^{M_1N_1 \ldots M_4N_4} t_{M'_1N'_1 \ldots M'_4N'_4} \hat{R}^{M'_1N'_1}{}_{M_1N_1} \ldots \hat{R}^{M'_4N'_4}{}_{M_4N_4} + \frac{1}{4} E_8 . \end{aligned} \tag{15}$$

The tensor t is defined by $t^{M_1 \ldots M_8} A_{M_1M_2} \ldots A_{M_7M_8} = 24 \mathrm{tr} A^4 - 6 (\mathrm{tr} A^2)^2$ for antisymmetric tensors A.[9] Note that E_8 given in (15) is not the eight-dimensional Euler density but an 11-dimensional generalization of it. More generally one can define [23]

$$\begin{aligned} E_n(M_D) &= \frac{1}{(D-n)!} \epsilon_{N_1 \ldots N_{D-n} N_{D-n+1} \ldots N_D} \epsilon^{N_1 \ldots N_{D-n} N'_{D-n+1} \ldots N'_D} \\ &\quad R^{N_{D-n+1} N_{D-n+2}}{}_{N'_{D-n+1} N'_{D-n+2}} \ldots R^{N_{D-1} N_D}{}_{N'_{D-1} N'_D} , \end{aligned} \tag{16}$$

[6] We have adapted the formula given in [2] to our conventions.

[7] Strictly speaking Δ could have a harmonic part. However, this should also vanish in the limit $l_{11}/l_Y \to 0$ in order to ensure that (12) becomes the unwarped metric.

[8] All other bosonic higher derivative terms which are related via supersymmetry to the ones given in (9) and (14) are proportional to the Ricci-tensor or contain at least one 4-form field strength [23]. Their contribution to the potential is therefore subleading.

[9] We follow here the conventions of [23] which differ from the tensor t_8 used in [24] in that the ϵ-term is omitted.

where D denotes the real dimension of the manifold. Then $E_8(Y_4)$ is proportional to the eight-dimensional Euler density, i.e. $12b_1 \int_{Y_4} d^8\xi\sqrt{g}\, E_8(Y_4) = \chi$ holds.

The next step is to reduce the action $\mathcal{S} + \delta\mathcal{S}_1 + \delta\mathcal{S}_2$ consisting of the terms given in (1), (9) and (14). Following ref. [2] we suppose that the only non-vanishing components of F_4 are F_{mnpq} and $F_{\mu\nu\rho m}$ where the latter are related via supersymmetry to the warp factor via $F_{\mu\nu\rho m} = \epsilon_{\mu\nu\rho}\partial_m e^{-\frac{3}{2}\Delta}$. In the reduction of the 11-dimensional Einstein-Hilbert term one obtains the three-dimensional Einstein-Hilbert term $R^{(3)}$ and an additional term proportional to $\int_{Y_4} \Delta\nabla_m\partial^m\Delta$ which is of higher order.[10] Furthermore, the integral $\int_{Y_4} d^8\xi\sqrt{g^{(8)}}J_0$ vanishes for Ricci-flat Kähler manifolds [25,26].[11] Again, at leading order we can assume the metric to be Ricci flat and neglect the effect of the warp factor in J_0 as a higher order contribution. To evaluate the leading order contribution to E_8 we use the fact that on a product space $M_3 \times Y_4$ one has $E_8(M_3 \times Y_4) = -E_8(Y_4) + 4E_2(M_3)E_6(Y_4)$ where $E_2(M_3) = -2R^{(3)}$ holds.

The details of the Kaluza-Klein reduction procedure can be found in [7,14] while here we only give a few intermediate steps. First of all one obtains a non-canonical Einstein term in the three-dimensional effective action

$$\mathcal{S}^{(3)} = \frac{1}{2\kappa_{11}^2}\int d^3x\sqrt{-g^{(3)}}\Lambda R^{(3)} + \dots\ , \tag{17}$$

where

$$\Lambda = \mathcal{V}_\Delta + 8\kappa_{11}^2 b_1 T_2 \int_{Y_4} d^8\xi\sqrt{g^{(8)}}E_6(Y_4)\ , \tag{18}$$

and $\mathcal{V}_\Delta = \int_{Y_4} d^8\xi e^{-\Delta/2}\sqrt{\hat{g}^{(8)}}$ denotes a warped Calabi-Yau volume, where $\hat{g}^{(8)}$ is the internal part of the warped metric (12). With the help of a Weyl rescaling $g_{\mu\nu} \longrightarrow \Lambda^2 g_{\mu\nu}$ the Einstein term can be put into canonical form. At leading order Λ can be replaced by $\mathcal{V}$ and one obtains the Weyl rescaled low energy effective Lagrangian (we set $\kappa_{11} = 1$ henceforth)[12]

$$\mathcal{L}^{(3)} \ = \ \mathcal{L}_0^{(3)} - \frac{1}{2}\,\epsilon^{\mu\nu\rho}\,\tilde{W}_{AB}A_\mu^A F_{\nu\rho}^B \ - V\ , \tag{19}$$

where $\mathcal{L}_0^{(3)}$ is given in (2) and one has

$$\begin{aligned} V &= \frac{1}{4\mathcal{V}^3}\left(\int_{Y_4} d^8\xi\sqrt{g^{(8)}}|F_4|^2 - \frac{1}{6}T_2\,\chi\right)\ , \\ \tilde{W}_{AB} &= \frac{1}{2}\partial_A\partial_B\tilde{W}\ , \qquad \tilde{W} = \frac{1}{4}\int_{Y_4} F_4\wedge J\wedge J\ . \end{aligned} \tag{20}$$

For $\chi = 0$ both V and the Chern-Simons terms were absent.[13]

In order to display the relationship of the potential with the two superpotentials of [9] we need to further rewrite V. By definition we have

$$\int_{Y_4} d^8\xi\sqrt{g^{(8)}}|F_4|^2 = \int_{Y_4} F_4\wedge\star F_4\ , \tag{21}$$

[10] The absence of possible other terms in the presence of a 4-form flux is carefully discussed in ref. [14].

[11] J_0 is the sum of an internal and an external part. Since J_0 can be expressed through the Weyl-tensor only [27,28] the external part vanishes because the Weyl tensor vanishes identically in $D = 3$.

[12] As we said before $\mathcal{L}_0^{(3)}$ is also corrected at this order but we did not compute those correction.

[13] The fact that the E_8-term contributes to the potential in $D = 3$ was first noticed in [29].

where to leading order $\star F_4$ is the Hodge dual of F_4 with respect to the metric g_{mn}. F_4 can be expanded as the sum $F_4 = F_{4,0} + F_{3,1} + F_{2,2} + F_{1,3} + F_{0,4}$. In order to proceed let us recall that on Y_4 a primitive (p,q)-form $\omega^{(0)}_{p,q}$ can be defined by

$$J^{5-p-q} \wedge \omega^{(0)}_{p,q} = 0 \ , \tag{22}$$

where J^n is the nth wedge product of the Kähler form. The Hodge dual of a primitive four-form is given by [9]

$$\star\omega^{(0)}_{p,4-p} = (-1)^p \omega^{(0)}_{p,4-p} \ . \tag{23}$$

On Y_4 all components of F_4 except $F_{(2,2)}$ are primitive in that they satisfy (22) and as a consequence their Hodge duals are simply given by

$$\star F_{4,0} = F_{4,0} \ , \qquad \star F_{3,1} = -F_{3,1} \ , \qquad \star F_{1,3} = -F_{1,3} \ , \qquad \star F_{0,4} = F_{0,4} \ . \tag{24}$$

For $F_{(2,2)}$ one uses the Lefschetz decomposition which asserts [31]

$$F_{2,2} \equiv F^{(0)}_{2,2} + J \wedge F^{(0)}_{1,1} + J^2 \wedge F^{(0)}_{0,0} \ , \tag{25}$$

and computes explicitly [14]

$$\star F_{2,2} = F^{(0)}_{2,2} - J \wedge F^{(0)}_{1,1} + J^2 \wedge F^{(0)}_{0,0} = F_{2,2} - 2J \wedge F^{(0)}_{1,1} \ . \tag{26}$$

Combining eqs. (24)–(26) one arrives at

$$\star F_4 = F_4 - 2F_{3,1} - 2F_{1,3} - 2J \wedge F^{(0)}_{1,1} \ , \tag{27}$$

and hence

$$\int_{Y_4} F_4 \wedge \star F_4 = \int_{Y_4} F_4 \wedge F_4 - 4 \int_{Y_4} F_{3,1} \wedge F_{1,3} - 2 \int_{Y_4} J \wedge F^{(0)}_{1,1} \wedge J \wedge F^{(0)}_{1,1} \ , \tag{28}$$

where we have used $J \wedge F^{(0)}_{2,2} = 0$ and $J^3 \wedge F^{(0)}_{1,1} = 0$ in accord with (22).

The second term in (28) can be further rewritten by using [16]

$$D_\alpha \Omega \equiv \partial_\alpha \Omega + (\partial_\alpha K_{3,1})\, \Omega = \Phi_\alpha \tag{29}$$

where Φ_α is the basis of $H^{3,1}$ and $K_{3,1}$ is the Kähler potential for the $(3,1)$ moduli defined in eq. (3). With the help of (29) and (3) one derives

$$\int_{Y_4} F_{3,1} \wedge F_{1,3} = -e^{K_{3,1}} G^{-1\alpha\bar\beta} D_\alpha W D_{\bar\beta} \bar W \ , \tag{30}$$

where

$$W = \int_{Y_4} \Omega \wedge F_4 \tag{31}$$

is precisely the chiral superpotential of [9].

Finally, with the help of (4) also the last term in (28) can be expressed in terms of the superpotential (20). Expanding $F^{(0)}_{1,1}$ into the basis e_A and using

$$G^{-1AB} = -\frac{1}{6}\mathcal{V}\mathcal{V}^{-1AB} + \frac{2}{3} M^A M^B \ , \tag{32}$$

where $\mathcal{V}_{AB} = \frac{1}{12}\partial_A\partial_B\mathcal{V}$ and indices are raised with δ^{AB}, one finds

$$\int_{Y_4} J \wedge F^{(0)}_{1,1} \wedge J \wedge F^{(0)}_{1,1} = -\mathcal{V}^{-1}\left(G^{-1AB}\partial_A\tilde{W}\partial_B\tilde{W} - 2\tilde{W}^2\right) . \tag{33}$$

Inserting (30), (33) into (28) and using (21) one arrives at

$$\begin{aligned}\int_{Y_4} d^8\xi\sqrt{g^{(8)}}|F_4|^2 \;&=\; \int_{Y_4} F_4 \wedge F_4 + 4e^{K_{3,1}}G^{-1\alpha\bar{\beta}}D_\alpha W D_{\bar{\beta}}\bar{W} \\ &\quad +2\ \mathcal{V}^{-1}\left(G^{-1AB}\partial_A\tilde{W}\partial_B\tilde{W} - 2\tilde{W}^2\right) .\end{aligned} \tag{34}$$

Inserting (34) into (20) and taking into account the tadpole cancellation condition (11) results in

$$V = e^{K^{(3)}}G^{-1\alpha\bar{\beta}}D_\alpha W D_{\bar{\beta}}\bar{W} + \mathcal{V}^{-4}\left(\frac{1}{2}G^{-1AB}\partial_A\tilde{W}\partial_B\tilde{W} - \tilde{W}^2\right) , \tag{35}$$

where

$$K^{(3)} = K_{3,1} - 3\ln\mathcal{V} . \tag{36}$$

Finally, V and $\mathcal{L}^{(3)}$ can be written in a more canonical form by transforming to new coordinates

$$\hat{M}^A = \mathcal{V}^{-1}M^A , \qquad \hat{J} = \hat{M}^A e_A . \tag{37}$$

From eqs. (4), (5) and (36) we learn

$$\hat{\mathcal{V}} \equiv \int_{Y_4} \hat{J}^4 = \mathcal{V}^{-3} , \qquad \hat{G}_{AB} = \mathcal{V}^2 G_{AB} = -\frac{1}{2}\partial_A\partial_B\ln\hat{\mathcal{V}} , \qquad K^{(3)} = K_{3,1} + \ln\hat{\mathcal{V}} , \tag{38}$$

where the derivatives ∂_A are now with respect to $\hat{M}^A$. If we furthermore introduce

$$\hat{W} = \frac{1}{4}\int_{Y_4} F_4 \wedge \hat{J} \wedge \hat{J} = \mathcal{V}^{-2}\tilde{W} \tag{39}$$

and insert (38) and (39) into (19) using (2) we arrive at

$$\begin{aligned}\mathcal{L}^{(3)} \;&=\; \frac{1}{2}R^{(3)} - G_{\alpha\bar{\beta}}\partial_\mu Z^\alpha\partial^\mu\bar{Z}^{\bar{\beta}} - \frac{1}{2}\hat{G}_{AB}\partial_\mu\hat{M}^A\partial^\mu\hat{M}^B - \frac{1}{4}\hat{G}_{AB}F^A_{\mu\nu}F^{B\mu\nu} \\ &\quad -\frac{1}{2}\,\epsilon^{\mu\nu\rho}\,\hat{W}_{AB}A^A_\mu F^B_{\nu\rho}\ - V ,\end{aligned} \tag{40}$$

where the explicit dependence on $\hat{\mathcal{V}}$ has disappeared from the Lagrangian. The potential in the new variables is given by

$$V = e^{K^{(3)}}G^{-1\alpha\bar{\beta}}D_\alpha W D_{\bar{\beta}}\bar{W} + \frac{1}{2}\hat{G}^{-1AB}\partial_A\hat{W}\partial_B\hat{W} - \hat{W}^2 , \tag{41}$$

where the derivatives $\partial_A\hat{W}$ are again taken with respect to the new variables $\hat{M}^A$.

We see that the potential is entirely expressed in terms of two superpotentials W and $\hat{W}$. The W given in (31) is precisely the chiral superpotential of [9] while $\hat{W}$ given in (39) is the real version of the superpotential of [9]. This is related to the fact that the presence of the Chern-Simons terms no longer allows a duality transformation from vector to chiral multiplets. As a consequence $\hat{W}$ can not be complexified as there are only real scalars $\hat{M}^A$ in the vector multiplets. However, upon further S^1 reduction $\hat{W}$ should become complex and coincide with the $D = 2$ superpotential of [11, 9].[14]

Finally, let us compare the potential (41) with the potentials of $D = 3, N = 2$ supergravity. Unfortunately, the relevant potential for chiral and vector multiplets with Chern-Simons terms coupled to $D = 3, N = 2$ supergravity is not available in the literature. The full derivation of this potential is beyond the scope of this paper and will be presented elsewhere. Here we just derive part of it by an S^1 compactification of a corresponding $D = 4$ supergravity.

The four-dimensional theory we need to consider has to contain both chiral and linear multiplets. A $D = 4$ linear multiplet consists of an antisymmetric tensor and a real scalar L as bosonic components. The $D = 4$ Lagrangian is determined by two functions, the holomorphic superpotential $W(\phi^\alpha)$ and the real function $K^{(4)} = K_\phi(\phi^\alpha, \bar{\phi}^{\bar{\alpha}}) + K_L(L^{\tilde{A}})$, where ϕ^α denote the scalars of the chiral multiplets. K_ϕ is the Kähler potential of the chiral fields and the second derivative of K_L determines the σ-model metric of $L^{\tilde{A}}$ [32, 33].[15] In this theory the scalar potential is given by[16]

$$V^{(4)} = e^{K^{(4)}} \left(G^{-1\alpha\bar{\beta}} D_\alpha W D_{\bar{\beta}} \bar{W} + (L^{\tilde{A}} \partial_{\tilde{A}} K^{(4)} - 3)|W|^2 \right) , \tag{42}$$

where $D_\alpha W = \partial_\alpha W + (\partial_\alpha K^{(4)})W$. This form of the potential can be derived for example from the $D = 4$ duality between an antisymmetric tensor and a scalar. At the level of superfields this results in the duality between a linear multiplet L and a chiral multiplet S with $S + \bar{S} \sim L^{-1}$. The Kähler potential of S is given by $K = -\ln(S + \bar{S})$ and the superpotential continues to be a function of only the ϕ^α. In this dual description with only chiral multiplets $V^{(4)}$ takes the standard form $V^{(4)} = e^{K^{(4)}} \left(G^{-1I\bar{J}} D_I W D_{\bar{J}} \bar{W} - 3|W|^2 \right)$, where the index I now runs over all chiral multiplets $(\phi^\alpha, S^{\tilde{A}})$.

Reducing the theory on a circle leaves the chiral multiplets unaltered. The linear multiplets, however, become vector multiplets and an additional vector multiplet containing the radius r and the Kaluza-Klein vector of the circle as its bosonic components appears in the spectrum. Let us define $L^0 = r^{-2}$. A straightforward S^1-reduction shows that after an appropriate Weyl rescaling the $D = 3$ potential is given by

$$V^{(3)} = e^{K^{(3)}} \left(G^{-1\alpha\bar{\beta}} D_\alpha W D_{\bar{\beta}} \bar{W} + (L^A \partial_A K^{(3)} - 4)|W|^2 \right) , \tag{43}$$

where $K^{(3)} = K^{(4)} + \ln L^0$ and the index A now includes 0. This form of the potential is indeed consistent with the first term of (41) if one identifies $L^A = \hat{M}^A$, uses (38) and the identity $\hat{M}^A \partial_A \ln \hat{\mathcal{V}} = 4$.

The second term in (41) is precisely the D-term of a $D = 3, N = 2$ Chern-Simons theory coupled to supergravity. As was noted in ref. [34] supersymmetrization of the Chern-Simons terms requires a D-term potential which coincides with the second term in

[14] To show this directly would require a generalization of the arguments given in [30]. The reduction of the Chern-Simons terms leads to terms $\sim \theta_A F^A_{01}$ and the θ-angles contribute to the potential in $D = 2$.

[15] We have chosen a theory where $K^{(4)}$ is the sum of two terms since this matches the situation we have in $D = 3$.

[16] We thank R. Grimm for discussions on this point.

(41). The last term of (41) should arise when not only a pure Chern-Simons theory but a more general gauge theory including the standard kinetic term and the Chern-Simons term is coupled to $N = 2$ supergravity.

Curiously, the potential for $\hat{W}$ as displayed in (41) is the $D = 3$ version of a general formula for the scalar potential in arbitrary dimension D. The requirement of the stability of AdS backgrounds leads to [35]

$$V^{(D)} = \frac{1}{2}(D-2)(D-1)\left(\frac{D-2}{D-1}G^{-1AB}\partial_A\hat{W}\partial_B\hat{W} - \hat{W}^2\right) , \tag{44}$$

which for $D = 3$ indeed gives $V^{(3)} = \frac{1}{2}G^{-1AB}\partial_A\hat{W}\partial_B\hat{W} - \hat{W}^2$.

Finally, let us discuss the conditions for unbroken supersymmetry. From our derivation of the $D = 3$ potential it is clear that for the chiral multiplets unbroken supersymmetry requires $D_\alpha W = 0$ which in turn imposes constraints on the allowed 4-form flux F_4. Using (31) $D_\alpha W = 0 = D_{\bar{\beta}}\bar{W}$ implies $F_{3,1} = F_{1,3} = 0$. Furthermore, from our derivation of the potential one would expect that for the vector multiplets the conditions of unbroken supersymmetry are[17]

$$(\partial_A K^{(3)})\, W \; = \; 0 \; = \; \partial_A \hat{W} \ , \tag{45}$$

while the vacuum energy is determined by $\hat{W}^2$ only. The first condition is the precise analog of the four-dimensional situation when the superpotential does not depend on a chiral scalar, in which case $D_S W = (\partial_S K)W$ holds and supersymmetry also forces $W = 0$. As we have seen the $D = 3$ vector multiplets are closely related to the $D = 4$ linear multiplets which precisely have the feature $D_S W = (\partial_S K)W$. $(\partial_A K^{(3)})W = 0$ in general can only be fulfilled if $W = 0$ which implies $F_{4,0} = F_{0,4} = 0$. Finally, using (39) $\partial_A\hat{W} = 0$ implies $\hat{J} \wedge F_4 = 0$, i.e. F_4 has to be primitive. This last condition implies not only $\partial_A\hat{W} = 0$ but also $\hat{W} = 0$. Thus the cosmological constant always vanishes in a supersymmetric minimum and no supersymmetric AdS_3 solution exists.[18] To summarize, supersymmetry constrains the 4-form flux to be a primitive $(2,2)$-form $F_4 = F^{(0)}_{2,2}$. This result has first been derived in [2] and has led to the proposal of the two superpotentials (31) and (a complex version of) (39) in ref. [9]. By performing a Kaluza-Klein reduction on a Calabi-Yau fourfold with 4-form flux turned on we have verified that to order $\mathcal{O}(\kappa_{11}^{-2/3})$ the potential can be expressed in terms of the superpotentials (31), (39) and reproduced the conditions for a supersymmetric ground state.

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[17]A rigorous derivation requires the computation of the fermionic supersymmetry transformations in an $N = 2$ supergravity background.

[18]Recently this has also been noticed in the revised version of [9]. We thank the authors for communicating their result prior to publication.

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